# On the Numerical Solution of the Characteristic Problem for one Quasi-Linear Equation

Giorgi Baghaturia
Muskhelishvili Institute of Computational Mathematics of the Georgian Technical University Tbilisi, Georgia
e-mail: gi.baghaturia@gtu.ge

Marine Menteshashvili
Muskhelishvili Institute of Computational Mathematics of the Georgian Technical University
Sokhumi State University
Tbilisi, Georgia
e-mail: m.menteshashvili@gtu.ge

https://doi.org/10.62343/csit.2024.13

***Abstract*** — **For linear differential equations solving the Goursat problem means to find a solution to the equation by given values on arcs of characteristics of different families. These arcs have one common point and their tangents are different at this point. As for nonlinear equations, families of characteristic curves depend on the sought solution and therefore are unknown in advance. For that reason, it is impossible to pose the characteristic problem (analogue of Goursat problem) for a nonlinear equation in the same way we do it for a linear one. In this paper we present a numerical method to solve one class of quasi-linear equations whose one of characteristics is straight line. One of the families of characteristics is completely determined, while the other depends on the first derivatives of unknown solution and thus is not determined in advance. The type of equation is hyperbolic with possible parabolic degeneracy and this fact should be taken into account so that the problem is posed correctly. For finding a numerical solution of the problem we propose an algorithm which is based on the well-known method of characteristics.**

***Keywords*** — **Quasi-linear hyperbolic equation, Characteristic Problem, Grid-characteristic method.**

## I. INTRODUCTION

As is known, for linear equations, Goursat problem consists in finding a solution to the equation by its values given on the arcs of characteristics of different families coming out from one point. For the numerical solution of such problems, an effective method is the grid-characteristic method. Nowadays, the grid-characteristic method is gaining popularity for linear hyperbolic systems of equations (see, for example, [1-2]). As for nonlinear equations, families of characteristic curves depend on the value of the unknown solution and therefore are not known in advance. For this reason, posing the Characteristic problem in the same way as is known for linear equations (see, for example, [3]) is impossible. We will consider the case when for a quasilinear equation it is still possible to formulate similar problem and at the same time, we will justify the numerical method.

In this paper, we consider a quasilinear equation, on the example of which the characteristic problem is correctly posed and we also introduce a numerical algorithm to solve the problem. Equations of this type were considered in papers [4–13, 15], in which initial, characteristic and initial-characteristic problems were studied.

In the plane of variables x, y, consider the following second order quasi-linear equation

$$Lu \equiv u_{xx} + (1 + u_x + u_y)u_{xy} + (u_x + u_y)u_{yy} = f(x, y, u_x, u_y). \quad (1)$$

It should not be classified as strictly hyperbolic equation, since the corresponding characteristic form degenerates and this degeneration depends on behavior of solutions of the equation. In particular, when the sum of derivatives of solution $u_x + u_y$ equals to one, the equation has parabolic degeneracy. Therefore (1) is of mixed hyperbolic-parabolic type.

## II. CHARACTERISTIC PROBLEM

In this section, we consider the equation

$$Lu = u_x + u_y - 1 \quad (2)$$

The differential characteristic relations of equation (2) have the following form:

$$\begin{cases} dy = dx, \\ dp + (p + q)dq - (p + q - 1)dx = 0, \end{cases} \quad (3)$$

$$\begin{cases} dy = (p + q)dx, \\ dp + dq - (p + q - 1)dx = 0, \end{cases} \quad (4)$$

where $p = u_x,\ q = u_y$ are the Monge notations.

Having studied these systems, we come to the conclusion that one family of characteristic curves corresponding to system (3), is completely defined and is given by straight lines

$$y - x = const. \quad (5)$$

Along this family, the characteristic invariant

$$(p + q - 1)e^{q-x} \quad (6)$$

retains a constant value.

As for the second family of characteristics, it is defined as follows

$$(p + q - 1)e^{p+q-1-y} = const, \quad (7)$$

along which the invariant

$$(p + q - 1)e^{-x} \quad (8)$$

remains constant.

As the structure of invariant (8) shows, the family of characteristic curves corresponding to system (4) depends on the first derivatives of the unknown solution and thus it is not determined in advance. Since the characteristic family (6) is completely defined, the condition on this characteristic can be specified in the same way as in the case of linear equations. In particular, along the straight line $y = x + c$ the value of the solution is specified. We also have to check that this value does not cause parabolic degeneration of the equation at some point.

On the segment $J_1$ of the characteristic curve $F_1(x) = x$, consider the condition

$$u|_{J_1} = \varphi(x),\ x \in [0, a],\ \varphi \in C^2[0, a] \quad (9)$$

At each point of the segment $J_1$ we can compute a derivative of the solution with respect to the characteristic direction. Formally we can write that

$$(u_x + u_y)|_{J_1} = u_x(x,x) + u_y(x,x) = \varphi'(x).$$

To avoid parabolic degeneracy, we require that

$$\varphi'(x) \neq 1, \; x \in [0, \alpha]. \qquad (10)$$

The characteristic lines corresponding to differential relations (4) are completely determined, as they are given by equation (6). Due to the fact that the sum $p + q$ included in (6) can be calculated at each point $J_1$, we can draw a characteristic curve through any point of this segment that corresponds to system (4), Including the curve that passes through the end of $J_1$, and is defined explicitly:

$$y = (\varphi'(0) - 1)(e^x - 1) + x \equiv F_2(x). \qquad (11)$$

As we see, from the given value of the solution on the segment $J_1$, the curve of family (4) is clearly determined. Consequently, it is possible to set the value of the solution on the arc $J_2$ of the characteristic curve (11):

$$u|_{J_2} = \psi(x), \; x \in [0, b], \qquad (12)$$

where $\psi \in C^2[0, b]$ is a given function. We should note, that characteristics $J_1, J_2$, constitute the support of the problem conditions.

Therefore, the problem can be formulated as follows:

*Characteristic problem:*

Find a regular solution to equation (2) along with its domain of definition, if characteristic conditions (9) and (12) are satisfied, when $\varphi(0) = \psi(0)$, $\varphi'(0) = \psi'(0)$, $\varphi''(0) = \psi''(0)$.

The following Theorem is true:

*Theorem 1:* If from the functional equation

$$y_0 = (\varphi'(x_1) - 1)(e^{x_0 - x_1} - 1) + x_0,$$

we can uniquely find the quantity $x_1$ as a function of variables $x_0, y_0$ there exists a unique regular solution of the Characteristic Problem (2), (9), (12), and this solution is defined in the Domain D, which is bounded by

$J_1$, $J_2$, $J_3$: $y = (\varphi'(a) - 1)(e^{x-a} - 1) + x$,

$J_4$: $y = (\varphi'(0) - 1)(e^b - 1) + x, \; x \in [b, a + b]$

characteristic curves.

*Proof:* We are able to find the values of the derivatives of the solution using the curves $J_1$ and $J_2$:

$$u_x|_{J_1} = \varphi'(x) - log\frac{\psi'(0) - 1}{\varphi'(x) - 1} + \frac{\psi'(0) - 1}{\varphi'(0) - 1} - x + 1,$$

$$u_y|_{J_1} = log\frac{\psi'(0) - 1}{\varphi'(x) - 1} + \frac{\psi'(0) - 1}{\varphi'(0) - 1} + x - 1,$$

$$u_x|_{J_2} = (\varphi'(0) - 1)e^x - \frac{\psi'(x) - 1}{\varphi'(0) - 1}e^{-x} + 2,$$

$$u_y|_{J_2} = \frac{\psi'(x) - 1}{\varphi'(0) - 1}e^{-x} - 1.$$

The domain of definition of the solution to the problem will be bounded by the characteristic curves emerging from the ends of the segment $J_1$ and the arc $J_2$. These curves and arcs of characteristics $J_1$, $J_2$, create the curvilinear quadrilateral $D$, which is completely covered by characteristic lines emanating from each point of the arc $J_1$ and $J_2$. For example, through the arbitrary point $(x_1, x_1)$ of the segment $J_1$ passes a characteristic, equation of which has the following form:

$$J_5: \; y = (\varphi'(x_1) - 1)(e^{x - x_1} - 1) + x,$$

And through the arbitrary point $(x_2, y_2)$ of the segment $J_2$ passes a characteristic curve given by the equation:

$$J_6: y = (\varphi'(0) - 1)(e^{x_2} - 1) + x.$$

For arbitrary chosen characteristics $J_5$, $J_6$ it's easy to determine both the point of intersection $(x^*, y^*)$ and the value of the solution at this point. The coordinates $x^*, y^*$ are determined as functions of $x_1, x_2$:

$$x^* = x_1 + log\left(1 + \frac{(\varphi'(0) - 1)(e^{x_2} - 1)}{\varphi'(x_1) - 1}\right),$$

$$y^* = x^* + (\varphi'(0) - 1)(e^{x_2} - 1)$$

Along the segment $J_5$ the following relation is true:

$$log(p + q - 1) - x = log(\varphi'(x_1) - 1) - x_1 \qquad (13)$$

Analogously, along the segment $J_6$ we have:

$$log(p + q - 1) + q - x =$$

$$=\log\left((\varphi'(0) - 1)e^{x_2}\right) + \frac{\psi'(x_2) - 1}{\varphi'(0) - 1}e^{-x_2} - 1 - x_2 \qquad (14)$$

Let us assume that the curves $J_5$ and $J_6$ intersect at point $(x_0, y_0)$. In this case, from the relations (13), (14), derivatives of the solution at this point are determined as follows:

$$p(x_0, y_0) = (\varphi'(x_1) - 1)\left(e^{x_0 - x_1} + 2 - log\frac{(\varphi'(0) - 1)e^{x_2}}{\varphi'(x_1) - 1} - \frac{\psi'(x_2) - 1}{\varphi'(0) - 1}e^{-x_2} - x_1 + x_2\right)$$

$$q(x_0, y_0) = log\frac{(\varphi'(0) - 1)e^{x_2}}{\varphi'(x_1) - 1} + \frac{\psi'(x_2) - 1}{\varphi'(0) - 1}e^{-x_2} + x_1 - x_2 - 1.$$

So, it appears that we can find the value of the solution at this point.

All these considerations do not mean that we constructed the solution in an explicit way. In order to do so, we should have a possibility to find a value of our solution dependent on $(x_0, y_0)$, at any point $(x_0, y_0) \in D$. Let's draw the characteristic curves of the both families at this point. Let's denote by $J_7$ a straight line passing through the point $(x_0, y_0)$, corresponding to the system (3). By $J_8$ we denote the characteristic which corresponds to the system (4).

Let us also denote by $(x_1, x_1)$ point of intersection of $J_8$ with the segment $J_1$. Thus, the equation of characteristic curve $J_8$ will be written as follows:

$$J_8: \; y = (\varphi'(x_1) - 1)(e^{x - x_1} - 1) + x$$

If we consider equations of $J_2$ and $J_7$ as a functional system, with respect of variables $x, y$, it is easy to determine the coordinates of their point of intersection:

$$x_2 = log\left(1 + \frac{y_0 - x_0}{\varphi'(0) - 1}\right),$$

$$y_2 = (\varphi'(0) - 1)(e^{x_2} - 1) + x_1.$$

We should find the coordinates of the point of intersection of characteristics $J_8$ and $J_1$. Since, the equation of the curve $J_8$ contains unknown quantity $x_1$, we have to act in a different way: According to our condition, $J_8$ goes through $(x_0, y_0)$. So, we can write:

$$y_0 = (\varphi'(x_1) - 1)(e^{x_0 - x_1} - 1) + x_0.$$

And from this relation we should determine $x_1$ as a function of variables $x_0, y_0$. □

## III. NUMERICAL METHOD

After these considerations, in order to solve the problem (2), (9), (12), numerically, we can formulate grid-characteristic method.

On the segment $[0, a]$ we introduce a grid $\omega_{h_1}$:

$$\omega_{h_1} = \left\{ (\widetilde{x_0^j}, \widetilde{y_0^j}), \quad \widetilde{x_0^j} = h_1 j, \quad \widetilde{y_0^j} = F_1\left(\widetilde{x_0^j}\right), \quad j = 1,2,\dots,n_1, \ h_1 = \frac{a}{n_1}, n_1 \in \mathrm{N} \right\}.$$

On the segment $[0, b]$ we introduce a grid $\omega_{h_2}$:

$$\omega_{h_2} = \{ (\widetilde{x_i^0}, \widetilde{y_i^0}), \ \widetilde{x_i^0} = h_2 i, \ \widetilde{y_i^0} = F_2(\widetilde{x_i^0}), \quad i = 1,2,\dots,n_2, \ h_2 = \frac{b}{n_2}, \ n_2 \in \mathrm{N} \}.$$

Points $(\widetilde{x_i^0}, \widetilde{y_i^0})$, $i = 1,2,\dots,n_2$ are the first row of calculated points. The next row of points is obtained by the intersection of characteristics of the family (3) and (4) coming from the points of first row and from point $(\widetilde{x_0^j}, \widetilde{y_0^j})$ respectively, etc. If the $j$-th row of designed points $(\widetilde{x_i^j}, \widetilde{y_i^j})$, is determined, then the first approximation of the next row is determined by the formulas:

$$\frac{\tilde{y}_{i+1}^{j+1} - \tilde{y}_i^{j+1}}{\tilde{x}_{i+1}^{j+1} - \tilde{x}_i^{j+1}} = 1, \tag{15}$$

$$\frac{\tilde{y}_{i+1}^{j+1} - \tilde{y}_{i+1}^{j}}{\tilde{x}_{i+1}^{j+1} - \tilde{x}_{i+1}^{j}} = \tilde{p}_{i+1}^j + \tilde{q}_{i+1}^j, \tag{16}$$

where $\tilde{p}_{i+1}^j$, $\tilde{q}_{i+1}^j$ are the value of the functions $p, q$ at the point $(\tilde{x}_{i+1}^j, \tilde{y}_{i+1}^j)$.

After this, the values of the derivatives $u_x, u_y$ and the solution $u$ at the point $(\tilde{x}_{i+1}^{j+1}, \tilde{y}_{i+1}^{j+1})$, $i = 1,2,\dots, n_2 - 1$, $j = 1,2,\dots, n_1 - 1$ are calculated using the formulas:

$$\tilde{p}_{i+1}^{j+1} - \tilde{p}_i^{j+1} + \left(\tilde{p}_i^{j+1} + \tilde{q}_i^{j+1}\right)\left(\tilde{q}_{i+1}^{j+1} - \tilde{q}_i^{j+1}\right) + \left(1 - \tilde{p}_i^{j+1} - \tilde{q}_i^{j+1}\right)\left(\tilde{x}_{i+1}^{j+1} - \tilde{x}_i^{j+1}\right) = 0, \tag{17}$$

$$\tilde{p}_{i+1}^{j+1} - \tilde{p}_i^{j+1} + \tilde{q}_{i+1}^{j+1} - \tilde{q}_i^{j+1} - \left(\tilde{p}_{i+1}^j + \tilde{q}_{i+1}^j - 1\right)\left(\tilde{x}_{i+1}^{j+1} - \tilde{x}_i^{j+1}\right) = 0, \tag{18}$$

$$\tilde{u}_{i+1}^{j+1} - \frac{\tilde{u}_i^{j+1} + \tilde{u}_{i+1}^j}{2} = \frac{1}{2}\left(\tilde{p}_{i+1}^j + \tilde{p}_i^{j+1} + \tilde{q}_{i+1}^j + \tilde{q}_i^{j+1}\right)\left(\tilde{x}_{i+1}^{j+1} - \tilde{x}_{i+1}^j\right) \tag{19}$$

To clarify the calculation formulas, we use recalculation of the point $(\tilde{x}_{i+1}^{j+1}, \tilde{y}_{i+1}^{j+1})$ and of the values of $\tilde{p}_{i+1}^{j+1}, \tilde{q}_{i+1}^{j+1}, \tilde{u}_{i+1}^{j+1}$. These formulas have the following form for all $i = 1,2,\dots, n_2 - 1$, $j = 1,2,\dots, n_1 - 1$:

$$\frac{\tilde{y}_{i+1}^{j+1} - \tilde{y}_{i+1}^{j}}{\tilde{x}_{i+1}^{j+1} - \tilde{x}_{i+1}^{j}} = (\tilde{p}_{i+1}^{j+1} + \tilde{q}_{i+1}^{j+1} + \tilde{p}_{i+1}^j + \tilde{q}_{i+1}^j)/2, \tag{20}$$

$$\tilde{p}_{i+1}^{j+1} - \tilde{p}_i^{j+1} + \frac{1}{2}(\tilde{p}_{i+1}^{j+1} + \tilde{q}_{i+1}^{j+1} + \tilde{p}_i^{j+1} + \tilde{q}_i^{j+1})\left(\tilde{q}_{i+1}^{j+1} - \tilde{q}_i^{j+1}\right) + \left(1 - \tfrac{1}{2}(\tilde{p}_{i+1}^{j+1} + \tilde{q}_{i+1}^{j+1} + \tilde{p}_i^{j+1} + \tilde{q}_i^{j+1})\right)\left(\tilde{x}_{i+1}^{j+1} - \tilde{x}_i^{j+1}\right) = 0, \tag{21}$$

$$\tilde{p}_{i+1}^{j+1} - \tilde{p}_i^{j+1} + \tilde{q}_{i+1}^{j+1} - \tilde{q}_i^{j+1} - \left(\frac{\tilde{p}_{i+1}^{j+1} + \tilde{q}_{i+1}^{j+1} + \tilde{p}_{i+1}^j + \tilde{q}_{i+1}^j}{2} - 1\right)\left(\tilde{x}_{i+1}^{j+1} - \tilde{x}_i^{j+1}\right) = 0, \tag{22}$$

$$\tilde{u}_{i+1}^{j+1} - \frac{\tilde{u}_i^{j+1} + \tilde{u}_{i+1}^j}{2} = \frac{1}{2}\left(\tilde{p}_{i+1}^j + \tilde{p}_i^{j+1} + \tilde{q}_{i+1}^j + \tilde{q}_i^{j+1}\right)\left(\tilde{x}_{i+1}^{j+1} - \tilde{x}_{i+1}^j\right). \tag{23}$$

*Theorem 2:* Let $u \in C^4(D)$ and condition (10) be satisfied, then scheme (15-23) converges to the solution of the Characteristic problem (2), (9), (12) and the rate of convergence of the difference schemes is $O(h^2)$, where $h = max\,(h_1, h_2)$.

**Proof.** Let $u \in C^{3,3}(D)$, $D$ be an area bounded by characteristics $J_k$, $k = 1,..,4$ . From the relations (15), (16), we obtain that

$$y_{i+1}^{j+1} - y_i^{j+1} - x_{i+1}^{j+1} - x_i^{j+1} = O\left(\left(h_1^{j+1}\right)^2\right) \equiv \psi_{1,i}^{j+1}, \tag{24}$$

$$y_{i+1}^{j+1} - y_{i+1}^j - (p_{i+1}^j + q_{i+1}^j)(x_{i+1}^{j+1} - x_{i+1}^j = O\left(\left(h_2^{i+1}\right)^2\right) \equiv \psi_{2,i+1}^j, \tag{25}$$

where

$$h_1^{j+1} = \max_i \left|x_{i+1}^{j+1} - x_i^{j+1}\right|, \ h_2^{i+1} = \max_i \left|x_{i+1}^{j+1} - x_{i+1}^j\right|,$$

$i = 1,2,\dots, n_2 - 1$, $j = 1,2,\dots, n_1 - 1$.

We introduce the following notations:

$$t_i^j = \tilde{y}_i^j - y_i^j, \ r_i^j = \tilde{x}_i^j - x_i^j, \ s_i^j = \tilde{p}_i^j - p_i^j,$$
$$g_i^j = \tilde{q}_i^j - q_i^j \quad v_i^j = \tilde{u}_i^j - u_i^j.$$

Inserting the values of $\tilde{x}_i^j$, $\tilde{y}_i^j$ into (15), (16) and taking in account the equalities (24), (25), we obtain:

$$t_{i+1}^{j+1} - r_{i+1}^{j+1} = t_i^{j+1} - r_i^{j+1} - \psi_{1,i}^{j+1} = -\sum_{k=0}^{i} \psi_{1,k}^{j+1},$$

$$t_{i+1}^{j+1} - (p_{i+1}^j + q_{i+1}^j) r_{i+1}^{j+1} = t_{i+1}^j - (p_{i+1}^j + q_{i+1}^j) r_{i+1}^j + \left(s_{i+1}^j + g_{i+1}^j\right)\left(x_{i+1}^{j+1} - x_{i+1}^j\right) - \psi_{2,i+1}^j = -\sum_{k=0}^{j} \psi_{1,i+1}^{k+1}.$$

Consequently, if the condition (10) holds, from the last equations we conclude:

$$\left|r_{i+1}^{j+1}\right| \le c_1 h, \ \left|t_{i+1}^{j+1}\right| \le c_2 h, \ c_1, c_2 > 0$$

where

$h = \max\left(h_1^{j+1}, h_2^{i+1}\right)$, $i = 1,2,\dots, n_2 - 1$, $j = 1,2,\dots, n_1 - 1$.

It is easy to obtain analogous estimations for $s, g, v$.

And for the process of recalculation, we come to conclusion that if the condition (10) holds, then the following equalities are true:

$$\|r\|_C = O(h^2), \|t\|_C = O(h^2), \quad \|s\|_C = O(h^2),$$
$$\|g\|_C = O(h^2), \ \|v\|_C = O(h^2). \ \square$$

## ACKNOWLEDGMENT

The work was partially supported by the European Union's HORIZON EUROPE Grant Project GAIN.

## REFERENCES


1. Guseva, E.K., Golubev, V.I. & Petrov, I.B. Linear, Quasi-Monotonic and Hybrid Grid-Characteristic Schemes for Hyperbolic Equations. Lobachevski J Math 44, 296–312 (2023). https://doi.org/10.1134/S1995080223010146
2. Favorskaya, A.V., Petrov, I.B. (2018). Grid-Characteristic Method. In: Favorskaya, A., Petrov, I. (eds) Innovations in Wave Processes Modelling and Decision Making. Smart Innovation, Systems and Technologies, vol 90. Springer, Cham. https://doi.org/10.1007/978-3-319-76201-2_5
3. Berezin and N.P. Zhidkov, Computing Methods, part 2, Addison-Wesley; 1st edition, 1965
4. M. Menteshashvili. Study of the nonlinear characteristic problem using the Riemann invariants. Proceedings of A. Razmadze Mathematical Institute, v. 160, 2012, pp. 113-120.
5. G. Baghaturia, M. Menteshashvili. The general integral of a quasilinear equation and its application to solving a nonlinear characteristic problem. Siberian Mathematical Journal, 2019, Vol. 60, No. 6, pp. 940–951. New York, USA. Pleiades Publishing, Inc. DOI: 10.1134/S0037446619060028
6. M. Menteshashvili. The Nonlinear Cauchy Problem with Solutions Defined in Domains with Gaps. Journal of Mathematical Sciences, 2015, Volume 206, Issue 4, pp. 413-423, Springer.
7. G. Baghaturia, M. Menteshashvili. Application of General Integral of Quasi-linear Equation to Solving of Non-linear Cauchy Problem. Bulletin of TICMI, Vol. 27, No. 2, 2023, pp. 59–65, TSU Publ., Tbilisi, ISSN 1512-0082.
8. J. Gvazava, G. Baghaturia, M. Menteshashvili. Cauchy Problem for a Quasi-Linear Hyperbolic Equation with Closed Support of Data. Journal of Mathematical Sciences, 2013, Volume 193, Issue 3, pp. 364-368, Springer.
9. R. Bitsadze and M. Menteshashvili. On one nonlinear variant of the nonlocal characteristic problem. Journal of Mathematical Sciences, 2015, Volume 206, Issue 4, pp 406-412, Springer.
10. G. Baghaturia. Some non-linear versions of hyperbolic problems for one quasi-linear equation of mixed type. Journal of Mathematical Sciences (Springer), 2015, Volume 208, Issue 6, pp 621-634.
11. M. Z. Menteshashvili, On the Cauchy problem with initial data on the unit circle. (Russian) Soobsh. Akad. Nauk Gruzii 148 (1993), No. 2, 190–192.
12. J. K. Gvazava, Nonlocal and initial problems for quasi-linear, non-strictly hyperbolic equations with general solutions represented by superposition of arbitrary functions. Georgian Math. J. 10 (2003), No. 4, 687–707.
13. 6. J. K. Gvazava, The mean value property for non-strictly hyperbolic second order quasi-linear equations and the nonlocal problems. Proc. A. Razmadze Math. Inst. 135 (2004), 79–96.
14. E. Goursat, Lecons sur l’integration des equations aux derivees partielles du second ordre. Hermann, Paris 2, 1898.
15. V. Kvaratskhelia and M. Menteshashvili, Some Nonlinear Version of a Nonlocal Problem and Its Discrete Analogy, 2019 Computer Science and Information Technologies (CSIT), Yerevan, Armenia, 2019, pp. 77-78, doi: 10.1109/CSITechnol.2019.8894943.