Analysis of Time Measures for the M/G/1 System in a Random Environment

Revaz Kakubava
Georgian technical university
Tbilisi, Georgia
e-mail: r.kakubava@gmail.com

Nani Salia
Georgian technical university
Tbilisi, Georgia
e-mail: nanisalia2@gmail.com

https://doi.org/10.62343/csit.2024.18

The paper discusses the M/G/1 System in a Random Environment . In the system, the server is influenced by a random environment, the latter being a simple birth-death process. An analytical stochastic model is constructed and studied in terms of operational calculus. Steady state Laplace transforms for distribution functions of time measures namely virtual waiting time, sojourn time are derived.

***Keywords* — Queueing system, random environment, time measures, Markov process, Laplace transform**

I. INTRODUCTION

The analysis and design of modern information and telecommunication systems require the development of queueing models that consider the specifics of control, management, and external actions such as preemptive and batch service. This includes taking into consideration a variety of aspects, such as parallel operations and processes, variations in parameters and characteristics of arrivals and service, failures, and environmental renewal. Analytical challenges in many practical cases are effectively addressed within the framework of queueing systems operating in a random environment [1-5]. This is particularly important in systems with situationally integrable resources, where resources can be pooled to solve specific tasks as situations arise. Situational integrability refers to the ability to combine resources of a queueing system to address specific tasks or groups of tasks as situations occur. This concept is fundamental in modern information and telecommunication networks, information processing centers including multiprocessor and multi-computing systems, switching node commutation units including special computing complexes, and multichannel data transmission channels. In such systems, the number of service facilities i.e channels for a given flow of customers changes randomly over time due to equipment failure, reallocation of server units for preemptive service, shutdowns for verification, diagnostics, maintenance, etc. Furthermore, the model can be generalized to include partial failures and other scenarios beyond those described in existing publications. This includes studying the intensity of server channels' failures and renewals, which may depend on whether the system is idle or busy, i.e., the parameters of the random environment change depending on the state of the queuing system.

The additional information about the modern situation can be found in the used literature which is given at the end of the topic [3-9]. In this paper we consider the M/G/1 queue random environment. The input flow intensity is λ. The server is influenced by a random environment i.e. a random process μ(t) with a set of states I. I is a set with elements I = {0,1,2}, and changes its state based on the state of the environment. In other words, if μ(t) = í ∈ I, í belongs to I set, then the server is in state í. State 0 is special, indicating that the server cannot begin service even if there are customers in the queue. When the system is busy, it operates in service cycles or sequences of cycles, with probabilistic characteristics independent of cycle types.

When the server is idle, the random environment μ(t) is modeled as a birth-death process with a set of states I ={0,1,2} and transition intensities $\alpha_í$ from í → í+1 where (í=0,1) and $\beta_í$ from í → í-1 (í=1,2).($\alpha_í$ sub í from state í transits in state í+1, where)When the system is empty, the birth-death process acts as a random environment, and when the system is busy, the random environment affects only the initial probabilistic characteristics of the service.

This process can describe sequences of failures and renewals in a multi-channel redundancy system, with the states of the process μ(t) defining the number of operative channels or the number of channels allotted for serving a special flow. In service cycles, the random environment μ(t) is disregarded, affecting only the initial characteristics of the queueing system. The behavior of the environment affects only the functions $H_{íj}(x)$, which is interpreted as follows: $H_{íj}(x) = \mathbb{P}$ {the service time of a customer is less than x and the state of the server or the process μ(t) at the end of the service time is j}, provided that the server was in the state í at the beginning of the service [1,2].

II. ANALYSIS OF TIME MEASURES

In this section we analyze the probability characteristics of the waiting time of the demand and the sojourn time in the system. The Laplace-Stieltjes transforms of the distribution functions of these random variables are found.

Denote by $P_i(t)$, i = 1,2, the probability that at the moment of time t the server is in the state i and there are no demands in the service system.

I want to point out $u_i(x,t)$ as follows $u_i(x,t) = \lim_{h\to 0} \mathbb{P}$ $\{x < U(t) < x + h$; after expiration U(t) (virtual waiting time) the server is in the state $i\}$

Now I would like to formulate theorem n1

Theorem 1: The functions P_1, P_2, u_1 and u_2 satisfies the following system of differential and integro-differential equations which you can see on the slide

$$\frac{dP_1(t)}{dt} = -(\alpha+\lambda+\mu)P_1(t) + (\alpha+\beta)P_2(t) + u_1(t,0); \quad (1)$$

$$\frac{dP_2(t)}{dt} = -(\alpha+\beta+\lambda)P_2(t) + \mu P_1(t) + u_2(t,0); \quad (2)$$

$$\frac{\partial u_1(x,t)}{\partial t} - \frac{\partial u_1(x,t)}{\partial x} =$$

$$-\lambda\, u_1(x,t) + \lambda\int_0^x u_1(y,t)h_{11}(x-y)dy + \int_0^x u_2(y,t)h_{21}(x-y)dy +$$

$$+\lambda P_1(t)h_{11}(x) + \lambda P_2(t)h_{21}(x) + \alpha P_1(t)\mu e^{-\mu x}; \quad (3)$$

$$\frac{\partial u_2(x,t)}{\partial t}-\frac{\partial u_2(x,t)}{\partial x}=$$

$$-\lambda u_2(x,t)+\lambda\int_0^x u_1(y,t)h_{12}(x-y)dy+\lambda\int_0^x u_2(y,t)h_{22}(x-y)dy+\lambda P_1(t)h_{12}(x)+\lambda P_2(t)h_{22}(x). \quad (4)$$

Now lets prove

Proof:

Let us consider an infinitesimal time interval $(t, t+h)$ and trace the behavior of the system in this interval. Conventional probabilistic reasoning leads to the following ratios: which you can see on the slide

$$P_1(t+h)=P_1(t)[1-(\alpha+\lambda+\mu)h]+(\alpha+\beta)P_2(t)h+u_1(0,t)+o(h)$$

$$P_2(t+h)=P_2(t)[1-(\alpha+\beta+\lambda)h]+\mu P_1(t)h+u_2(0,t)h+o(h)$$

$$u_1(x,t+h)=u_1(x+h,t)(1-\lambda h)+\lambda h\int_0^x u_1(t,y)h_{11}(x-y)dy+\lambda h\int_0^x u_2(y,t)h_{21}(x-y)dy+\lambda hP_1(t)h_{11}(x)+\lambda hP_2(t)h_{21}(x)+\alpha hP_1(t)\mu e^{-\mu x}+o(h);$$

$$u_2(x,t+h)=u_2(x+h,t)(1-\lambda h)+\lambda h\int_0^x u_1(y,t)h_{12}(x-y)dy+\lambda h\int_0^x u_2(y,t)h_{22}(x-y)dy+\lambda hP_1(t)h_{12}(x)+\lambda hP_2(t)h_{22}(x)+o(h),$$

where $h_{ij}(x)=H'_{ij}(x)$. derivative

After simple transformations and passing to the limit at $h\to 0$, (h tends to 0) we obtain a system (1), (2), (3) and (4) expression.

We investigate the system at $t\to\infty$, (t tends to infinity) which corresponds to the stationary state.

The limits are denoted by $P_i=\lim_{t\to\infty}P_i(t)$; $u_i(x)=\lim_{t\to\infty}u_i(x,t)$; i=1,2, We assume these limits exist. Keep in mind that these limits are equal to 0.

$$\lim_{t\to\infty}\frac{p_1(t)}{dt}=0 \qquad \lim_{t\to\infty}\frac{\partial u_i(x,t)}{\partial t}=0$$

After passing to the limit at $t\to 0$ we obtain the following system:

$$(\alpha+\lambda+\mu)P_1=(\alpha+\beta)P_2+u_1(0); \quad (5)$$

$$(\alpha+\beta+\lambda)P_2=\mu P_1+u_2(0); \quad (6)$$

$$\frac{du_1(x)}{dx}=-\lambda u_1(x)-\lambda\int_0^x u_1(y)h_{11}(x-y)dy-\lambda\int_0^x u_2(y)h_{21}(x-y)dy-\lambda P_1(t)h_{11}(x)-\lambda P_2(t)h_{21}(x)-\alpha\mu P_1(t)e^{-\mu x}; \quad (7)$$

$$\frac{du_2(x)}{dx}=\lambda u_2(x)-\lambda\int_0^x u_1(y)h_{12}(x-y)dy-\lambda\int_0^x u_2(y)h_{22}(x-y)dy-\lambda P_1(t)h_{12}(x)-\lambda P_2(t)h_{22}(x) \quad (8)$$

Let us apply the Laplace transform to the last two equations. If we take into account the next expression which you see on the slide,

$$\int_0^\infty e^{-sx}\frac{du_1(x)}{dx}dx=s\int_0^\infty e^{-sx}u_1(x)dx-u_1(0),\ i=1,2$$

expressing $u_i(0)$ through $\mathbb{P}_i$, from first two equations (5), (6) and solving the system after transformation with respect to $u_i^*(s)$ we obtain the next expression:

$$u_i^*(s)=\frac{d_i(s)}{d(s)},\ i=1,2 \quad (9)$$

where

$d(s)=[s-\lambda+\lambda h_{11}^*(s)][s-\lambda+\lambda h_{22}^*(s)]-\lambda^2h_{12}^*(s)\cdot h_{21}^*(s)$;

$d_i(s)=[s-\lambda+\lambda h_{jj}^*(s)]\varphi_j(s)-\lambda h_{ji}^*(s)\varphi_i(s)$, $i+j=3$;

$\varphi_1(s)=(\alpha+\lambda+\mu)P_2-\mu P_1-\lambda P_1h_{12}^*(s)-\lambda P_2h_{22}^*(s)$;

$\varphi_2(s)=(\alpha+\beta+\lambda)P_1-(\alpha+\beta)P_2-\lambda P_1h_{11}^*(s)-\lambda P_2h_{21}^*(s)$-
$-\alpha\mu P_1/s+\mu$.

The expression for $u_i^*(s)$ contains two unknowns P_1 and P_2. To determine them we use the normalization condition

$$P_1+P_2+\int_0^\infty[u_1(x)+u_2(x)]dx=1,$$

or in operational form which you can see on expression No 10.

$$P_1+P_2+u_1^*(0)+u_2^*(0)=1 \quad (10)$$

Using (9), we obtain the expressions for $u_i^*(0)$, i=1,2. After substituting these expressions into (10), we have the following equation

$$[h_{12}^*(0)+h_{21}^*(0)\left(1+\frac{\alpha}{\mu}\right)-\mu(\tau_1-\tau_2)]+$$
$$+[h_{12}^*(0)+h_{21}^*(0)+(\alpha+\beta)(\tau_1-\tau_2]h_2=$$
$$=h_{12}^*(0)(1-\lambda\tau_2)+h_{21}^*(0)(1-\lambda\tau_1) \quad (11)$$

Here

$$\tau_1=\int_0^\infty x[h_{i1}(x)+h_{i2}(x)]dx=-[h_{i1}^*(s)+h_{i2}^*(s)]'_{s=0}.$$

i=1,2, i.e. τ_i is equal to the average service time per demand, assuming that service starts when the server is in the state i.

As it can be seen from (11), for the existence of the stationary state it is necessary to satisfy the condition which you can see on expression no 12.

$$h_{12}^*(0)(1-\lambda\tau_2)+h_{21}^*(0)(1-\tau_1)>0, \quad (12)$$

i.e. $d'(0)<0$.

To find the second equation with respect to P_1 and P_2 it is necessary to prove the following theorem.

Theorem 2: If condition (12) holds, the equation $d(s)=0$ has a real root $s_0>0$.

Proof: Consider a function $y=d(s)$ for real s. By direct substitution we obtain that d(0)=0. Moreover, as we pointed out above, condition (8) is equivalent to the condition d′<(0). It is also easy to see that $d(s)\to\infty$ at $s\to\infty$. Since $d(0)$=0 and $d'(0)<0$, in some neighborhood of point 0 the function y=d(s) decreases and, considering that d(∞) = ∞, it becomes obvious that there exists $s_0>0$ such that $d(s_0)=0$.

To illustrate this statement, let us show one of the variants for the graph of the function y=$d(s)$ in Figure. 1.

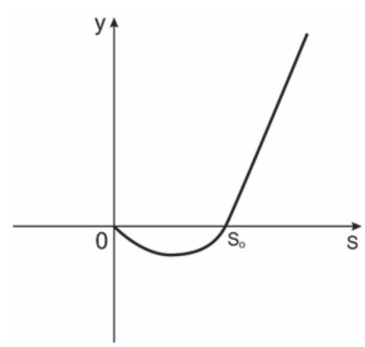

Fig.1

The functions $u_i^*(s)$ are analytic functions in the domain $Res>0$. Therefore, at the point s_0, $d_i(s_0)=0$, i=1,2. These equations give a single equation with respect to P_1 and P_2. It has the form $P_2=aP_1$,

where

$$a=\frac{[s_o-\lambda+\lambda h_{22}(s_o)]\left[s_o+\alpha+\mu-\frac{\alpha\mu}{s_o+\mu}\right]+\lambda\mu h_{21}^*(s_o)}{(\alpha+\beta)[s_o-\lambda+\lambda h_{22}^*(s_o)]+\lambda(s_o+\alpha+\beta)h_{21}^*(s_o)}.$$

Finally we obtain the expressions for P_1 and P_2:

$$P_1 = \frac{h_{12}^*(0)(1-\lambda\tau_2) + h_{21}^*(0)(1-\lambda\tau_1)}{[h_{12}^*(0)+h_{21}^*(0)]\left(1+\alpha+\frac{\alpha}{\mu}\right)+[(\alpha+\beta)\alpha-\mu][\tau_1+\tau_2]}$$

$$P_2 = aP_1 \qquad (13)$$

Let us denote by $\Phi(s)$ Laplace transform of the virtual waiting time distribution function for $U(t)$. It is obvious that if there are no demands in the system and $\mu(t)=0$, then $U(t)=0$ (the probability of this event is equal to $P_1 + P_2$), otherwise $U(t)>0$ (the probability density function of the virtual waiting time in this case is equal to $u_1(x)+u_2(x)$).

Hence

$$\Phi(s)=Me^{-s\omega(t)}= P_1 + P_2 + u_1^*(s)+u_2^*(s) \qquad (14)$$

Let us denote by $\Phi_1(s)$ the Laplace–Stieltjes transform of the distribution function for the customer sojourn time in the system. This time is the sum of waiting time and subsequent service time. The second term depends on the state in which the set of service devices was at the end of the waiting time (at the time of the beginning of subsequent service).

Considering the above, the following expression is obtained

$$\Phi_1(s)=[P_1 + u_1^*(s)]\, h_1^*(s)+[P_1+u_2^*(s)]h_2^*(s), \qquad (15)$$

where

$$h_i^*(s)=h_{i1}^*(s)+ h_{12}^*(s)$$

I would like to underline the fact that this work was supported by Shota Rustaveli National Science Foundation of Georgia.